A Case Study in Intersection Theory

Contents

1 Introduction

1.1 Motivation

Intersection theory lies at the heart of algebraic geometry. It has been around for centuries (in fact, millenniums!), as the problems it deals with – known as *enumerative* problems – are purely geometric. For instance, there is a famous problem in intersection theory that asks to find the circles in the plane tangent to three given ones. This problem, usually known as Appolloniu's problem in name of Apollonius de Perga (c. 252 BC - c. 190 BC), who posed and solved it, dates back to the Ancient Greek times, over 2000 years ago. It is remarkable that some problems like this were not just posed but also solved by these ancient mathematicians, who had no idea of the whole algebraic machinery we are about to present in this essay.

The only nature of this type of problems reveals some initial insights. First of all, it looks like the answer depends a lot on the circles we consider. Without thinking too much we can see there are many different initial configurations: the three circles might be disjoint, one might meet the other two or they might all meet. Even when they meet, their intersection can consist of one or two points, depending on whether the circles are tangent or not. More importantly, it looks like the answer to the problem will indeed depend on which of these cases we consider. One of the major insights in algebraic geometry in the 1980's was the realization that there is an answer that happens more frequently than all the others. It works for almost all cases, and finding what is the solution to these is actually the right question to ask. Thus, when dealing with enumerative problems we will always consider the *general* case, that is, the one that happens *most* of the times. Second, there are many enumerative problems that cannot be solved only by geometric arguments. and even the ones that can be solved often rely on exploiting particular features of the problem. This makes such solutions lack a generalization to similar problems, and they do not provide a systematic approach to solve enumerative problems. In [1] William Fulton developed a rigorous mathematical framework for these problems, giving a solid foundation to answer enumerative problems and starting modern intersection theory. As many other times in mathematics, algebra came into play to aid the intuitive ideas of geometry that characterize classical mathematics.

Throughout this essay I will try to show some of the power of this algebraic theory in enumerative geometry, but always trying to keep in mind that what we are doing, what we are solving, is pure geometry.

1.2 Outline

The main objective will be to introduce the algebraic machinery of the *Chow ring* to solve a few concrete problems in intersection theory. The focus will not be on developing such theory from the bottom but rather specifically for the Grassmannian through some particular examples. To show the power of the Chow ring, we have chosen examples that can also be solved by using only geometric arguments. In the process we will learn how to work with *Chern classes*, which will become extremely useful when dealing with problems like the number of lines contained on a cubic hypersurface in projective space.

The structure of the essay is as follows. We will start in Section 1.3 by presenting some basic results in algebraic geometry. In Section 2 we will develop the very basics of intersection theory, focusing on the definition of the Chow ring and some techniques to calculate it. Section 3 will be focused on calculating the Chow ring of our main object of study, the *Grassmannian* $G(k, n)$ parametrizing k-subspaces in an n-dimensional vector space, which gives rise to *Schubert calculus* [2]. We will continue in Section 4 introducing the theory of Chern classes, giving a geometric motivation for them and an axiomatic definition with the properties they satisfy. The most important will be Section 5. Here we will solve two classical problems both by geometric and algebraic reasoning, where the power of Schubert calculus and Chern classes will become clear.

1.3 Basic results in algebraic geometry

We remark that throughout this whole essay we will always consider an algebraically closed field of characteristic 0, having $\mathbb{C}$ as our model example. We start by stating four basic algebraic geometry results that we will need. For proofs or further exploration we refer to [3, 4].

The first result deals with the number of roots of a homogenous polynomial in $\mathbb{P}^1$. It will become important when proving Theorem 5.2.3.

Proposition 1.3.1. *Let $F(U, V)$ be a nonzero homogenous polynomial of degree d. Then, provided they are counted with multiplicities, F has exactly d zeros on $\mathbb{P}^1$.*

Note this is just a consequence (or rather a generalization) of the fundamental theorem of algebra, as restricting the polynomial to an affine patch gives a polynomial in $k[X]$.

The second result will give us a way to conclude two varieties are equal by checking only one inclusion.

Theorem 1.3.2. *If $X \subseteq Y$ then $\dim X \leq \dim Y$. Furthermore, if Y is irreducible and $X \subseteq Y$ is a closed subvariety with $\dim X = \dim Y$, then $X = Y$.*

This result is similar in flavor to the standard result in linear algebra asserting that for linear subspaces U and V with $\dim U = \dim V$, if $U \subset V$ then $U = V$. Note that an additional hypothesis regarding irreducibility is needed. We will use this result to prove Proposition 5.2.4.

The next two results deal with surjective regular maps and their fibers. They will be useful when working with *incidence varieties* and their projections to each of their factors.

Theorem 1.3.3. *Let $\pi : X \to Y$ be a surjective regular map between irreducible varieties. Then, for any $y \in Y$ and any component F of the fiber $\pi^{-1}(y)$, we have*

$$\dim F \geq \dim X - \dim Y.$$

Furthermore, there is an open subset $\emptyset \neq U \subseteq Y$ such that for every $y \in U$

$$\dim \pi^{-1}(y) = \dim X - \dim Y.$$

We see that the dimension of a fiber is constant in an open set, and open sets are dense in the Zariski topology. Thus, the dimension of the fiber is constant at most points, but it might vary in closed sets. This is usually referred to as *semi-continuity* of the dimension of the fiber.

As a consequence of the previous result, we obtain a useful criterion for irreducibility:

Theorem 1.3.4. *Let $\pi : X \to Y$ be a surjective regular map between projective varieties. If Y is irreducible and all the fibers $\pi^{-1}(y)$ for $y \in Y$ are irreducible and of the same dimension, then X is irreducible.*

This result will become useful in Section 5.2 when proving that certain incidence varieties are irreducible: we will find suitable projections to irreducible varieties and conclude with this theorem. Note the statement is very intuitive geometrically: if we attach an irreducible fiber to every point in an irreducible object, then the resulting object is necessarily irreducible. The purpose of surjectivity is to remove the possibility of having empty fibers or fibering over just two points. The constant dimension of the fiber is a bit more technical but nonetheless necessary (a counterexample is the incidence variety of lines tangent to the nodal cubic).

2 Introduction to intersection theory

In this section we will introduce the basics of intersection theory, including the construction of the Chow group, its ring structure and some techniques to calculate it. Further content can be found in [1, 5].

2.1 The Chow group of a variety

The Chow group of a variety is named after the Chinese mathematician Wei-Liang Chow. It is the algebro-geometric analogous of the homology of a topological space: it is formed out of subvarieties and these are identified by rational equivalence, just as singular homology is formed out of singular simplices and these are identified by “boundary equivalence”.

Given an algebraic variety X, we define the *group of cycles* on X, denoted as $Z(X)$, to be the free abelian group generated by all the subvarieties of X. This group is obviously graded by dimension, i.e.

$$Z(X) = \bigoplus_{k=0}^{\dim X} Z_k(X),$$

being $Z_k(X)$ the free abelian group generated by all the k-subvarieties of X. We will also use the notation $Z^k(X)$ for the free abelian group generated by the codimension k subvarieties of X, so that $Z_k(X) = Z^{\dim X - k}(X)$.

As just defined, this group vast. The natural way to reduce it is to declare two subvarieties to be the same if one can be deformed into the other. This is formalized by tge concept of *rational equivalence*:

Definition 2.1.1. We define $\mathrm{Rat}(X) \subset Z(X)$ to be the subgroup of cycles generated by elements of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

with $\Phi \subset \mathbb{P}^1 \times X$ a subvariety not contained in any fiber $\{t\} \times \mathbb{P}^1$. We say that two cycles $Z_1, Z_2 \in Z(X)$ are *rationally equivalent* if $Z_1 - Z_2 \in \mathrm{Rat}(X)$.

Intuitively, we are saying that two cycles are rationally equivalent if they can be deformed into each other through a family of subvarieties parametrized by $\mathbb{P}^1$ (see Figure 2.1). For example, all degree d hypersurfaces in projective space are rationally equivalent, just like any two fibers of a morphism to $\mathbb{P}^1$.

Definition 2.1.2. The *Chow group* of X is the quotient

$$A(X) = \frac{Z(X)}{\mathrm{Rat}(X)}.$$

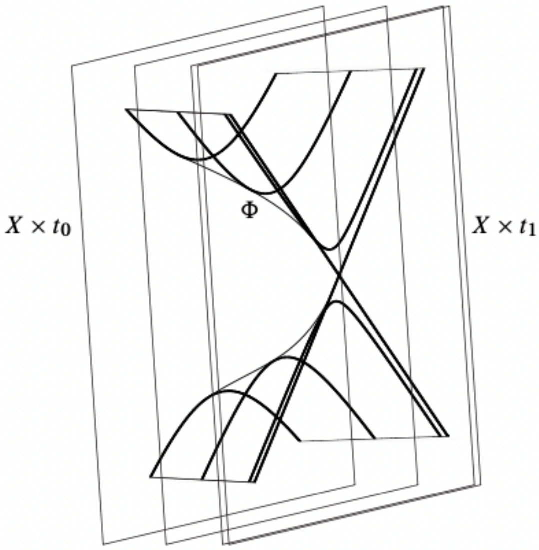

Figure 1: Rational equivalence between a hyperbola and the union of two lines in $\mathbb{P}^2$. Source: [5]

Given a cycle $Z \in Z(X)$, we will write $[Z]$ for its equivalence class. Therefore, given a subvariety $Y \subset X$, the class $[Y] \in A(X)$ will represent Y and any other subvariety rationally equivalent to Y.

2.2 Multiplicative structure

We have thus far discussed the group structure on the Chow group of X. When X is a smooth quasi-projective variety, we can endow the Chow group with a ring structure, and choosing appropiate cycles this product resembles their intersection. For this to be true the varieties need to intersect dimensionally "as expected": if $Z_1 \subset X$ is defined by k_1 equations and $Z_2 \subset X$ by k_2 equations, then we expect the intersection $Y_1 \cap Y_2$ to be defined by $k_1 + k_2$ equations. For this to be true, the equations must be suitably independent. The most natural way to ensure this independence is to ask for linear independence, and this is precisely what *transversality* does.

Near any smooth point $p \in Z_1 \cap Z_2$, the linearisation of X is T_pX and the linearisation of Z_i is $T_pZ_i \subset T_pX$. If we linearise the k_i equations defining Z_i we obtain a basis of the space of linear equations that define T_pZ_i in T_pX; in other words, a basis of the annihilator $(T_pZ_i)^\circ$ of T_pZ_i inside the dual space $(T_pX)^\vee$. The condition that the linear equations for Z_1 and Z_2 are independent then says that $(T_pZ_1)^\circ \cap (T_pZ_2)^\circ = 0$, which is equivalent to $T_pZ_1 + T_pZ_2 = T_pX$. We have arrived to the following definition:

Definition 2.2.1. We say that subvarieties $A, B \subset X$ *intersect transversely* at a point $p \in A \cap B$ if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X; that is,

$$T_pA + T_pB = T_pX,$$

or equivalently

$$\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim} T_pA + \operatorname{codim} T_pB.$$

We extend this definition to cycles $A = \sum A_i$ and $B = \sum B_i$ by declaring that they are transverse if A_i is transverse to B_j for all i and j.

For example, a proper subvariety will only be transverse to a point if it does not contain it, two different hyperplanes are always transverse, and any non-intersecting subvarieties are vacuously transverse.

As we will see in Theorem 2.2.4, this is actually stronger than what we need. Namely, we do not need transversality at *every* point, but rather at *most* points. To make precise sense of this idea, we say that a property holds *generically* if it holds for a dense subset of points. Then we define the following more general notion of transversality:

Definition 2.2.2. We say A and B are *generically transverse*, or that they *intersect generically transversely*, if they meet transversely at a general point of each component C of $A \cap B$.

Remark 2.2.3. Note that the set of points in which A and B intersect transverselly is open.

We are now in a position to state a fundamental result in intersection theory:

Theorem 2.2.4. *If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the condition:*

"If two subvarieties A, B of X are generically transverse, then

$$[A][B] = [A \cap B]."$$

This structure makes

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

into an associative, commutative ring graded by codimension.

Definition 2.2.5. We call $A(X)$, together with this unique ring structure, the *Chow ring* of X.

This result allows us to translate the geometric concept of intersection into a precise algebraic fact in the Chow ring. We will use it for the particular case of the Grassmannian, which as we will prove in Section 3 is a projective (and thus quasi-projective) smooth variety. The proof of the previous theorem comes historically from the following Lemma:

Theorem 2.2.6 (Moving lemma)**.** *Let X be a smooth quasi-projective variety.*

1. *For every $\alpha, \beta \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ with $[A] = \alpha$ and $[B] = \beta$.*
2. *The class $[A \cap B]$ is independent of the choice of such cycles A and B.*

Note that even if α and β are classes of a irreducible subvarieties, the cycles A and B guaranteed by the moving lemma might involve linear combinations, and in fact these might not even be effective. For instance, if $X = \mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^n$ and $\alpha = [E] = \beta$ are the class of the exceptional divisor E, then to get transversality we have to take cycles representing α of the form $H - (H - E)$, being H the total transform of a hyperplane in $\mathbb{P}^n$. As we move this hyperplane in its equivalence class, the cycles H and $H - E$ will also move in X. Choosing H disjoint from E and $H - E$ to be the strict transform of a hyperplane through the point we blew up we obtain transversality.

2.3 Building the Chow group

We now state a version of the so called Kleiman's theorem simplified to our setting (see [5] for full generality). Kleiman's theorem says that if we have a transitive action of a group on a variety, we can use this action to obtain general transversality. Additionally, it guarantees that the action preserves classes in the Chow ring. It is only valid for characteristic 0, but this will be enough for us as we will always work over $\mathbb{C}$.

Theorem 2.3.1 (Kleiman's theorem in characteristic 0)**.** *Suppose that an algebraic affine group G acts transitively on a variety X and that $A \subset X$ is a subvariety.*

1. *If $B \subset X$ is another subvariety, then there is an open dense set of elements $g \in G$ such that gA is generically transverse to B.*
2. *If G is affine, then $[gA] = [A] \in A(X)$ for any $g \in G$.*

Kleiman's theorem is of crucial importance when dealing with intersection theory problems in the Grassmannian, as the affine group GL_n has an obvious transitive action on the Grassmannian. We will use the first part very often two ensure we can assume general transversality, and the second to prove that some choices do not matter, such as when defining Schubert classes.

Lastly, we present a useful result to generate the Chow ring as a group. It relies on the concept of an *affine stratification*:

Definition 2.3.2. An *affine stratification* of a projective variety is a finite collection of quasi-projective subvarieties $U_i \cong \mathbb{A}^{k_i}$ of X such that:

1. X is a disjoint union of the U_i.
2. The closure of any U_i is a union of U_j.

Theorem 2.3.3. *If a projective variety X has an affine stratification, then $A(X)$ is generated by the classes of the closed strata.*

For example, the sequence $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ gives an affine stratification of $\mathbb{P}^n$ with closed strata the $\mathbb{P}^i$ and open open strata $\mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$. Thus, the graded abelian group $A(\mathbb{P}^n) = \oplus_{i=0}^n A_i(\mathbb{P}^n)$ has one generator in each degree, namely the classes of its linear subspaces $\mathbb{P}^i \subset \mathbb{P}^n$.

3 Intersection theory in the Grassmannian

This section will be devoted to introduce the Grassmannian $G(k, n)$, which parametrizes linear subspaces of dimension k in an n-dimensional vector space. Equivalently, the Grassmannian $\mathbb{G}(k-1, n-1)$ will parametrize $(k-1)$-linear subspaces of an $(n-1)$-dimensional projective space. We will show it can be embedded into projective space and that it is an algebraic variety. Lastly, we find an affine stratification and use Theorem to calculate its Chow ring. For this section we will follow [2] and [5].

3.1 Introduction to the Grassmannian

Definition 3.1.1. Let V be a vector space of dimension n and let $k \leq n$. The *Grassmannian* $G(k, V)$ is the set of k-dimensional linear subspaces of V. We will usually write $G(k, n)$ when the vector space is either clear or irrelevant.

Alternatively, we will denote by $\mathbb{G}(k, \mathbb{P}(V))$ (similarly, $\mathbb{G}(k, n-1)$) the set of linear subspaces of dimension k of the projective space $\mathbb{P}(V)$.

By definition, we recognize the projective space $\mathbb{P}^n$ as the Grassmannian $G(1, n+1)$. Furthermore, it is clear that there is a natural identification

$$G(k, n) \cong \mathbb{G}(k-1, n-1)$$

that takes each linear subspace to its projectivization. Thus, every result result valid for $G(k, n)$ will also be valid for $\mathbb{G}(k-1, n-1)$ and viceversa. We will work with the object that is more appropriate for each reasoning.

The first thing we will do is assign coordinates to $G(k, n)$. Let $W \in G(k, n)$ be a k-dimensional subspace of a given vector space V of dimension n. Choose a basis $e_1, \dots, e_n$ for V and a basis $w_1, \dots, w_k$ for W. Each w_i can be expressed as a sum

$$w_i = \sum_{i=1}^{n} \lambda_{ij} e_j, \quad \lambda_{ij} \in k.$$

Thus, with respect to these basis we can represent W by the $k \times n$ matrix

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k1} & \lambda_{k2} & \dots & \lambda_{kn} \end{pmatrix}.$$

The fact that the w_i are linearly independent means precisely that there exists a $k \times k$ minor with non-zero determinant. That is, if $1 \leq i_1 < \cdots < i_k \leq n$ and

$$p_{i_1 \dots i_k} := \begin{vmatrix} \lambda_{1i_1} & \lambda_{1i_2} & \dots & \lambda_{1i_k} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{ki_1} & \lambda_{ki_2} & \dots & \lambda_{ki_k} \end{vmatrix},$$

then the vector $(p_{i_1 \dots i_k})_{1 \leq i_1 < \cdots < i_k \leq n}$ is non-zero. Furthermore, if we choose another basis $w'_1, \dots, w'_k$ for W, the corresponding matrix B will satisfy $B = CA$ with C a non-singular matrix given by the change of basis. In particular, the new coordinates $p'_{i_1 \dots i_k}$ with respect to this basis will be

$$p'_{i_1 \dots i_k} = |C| \, p_{i_1 \dots i_k}.$$

so that $p'_{i_1 \dots i_k}$ and $p_{i_1 \dots i_k}$ differ only by a global factor. These two facts (non-zero vector and good definition up to scalars) suggest considering the map

$$G(k, n) \to \mathbb{P}^N$$

that assigns each k-dimensional subspace the point in $\mathbb{P}^N$ with coordinates $[p_{i_1, \dots, i_k}]$, where $N = \binom{n}{k} - 1$. We call such coordinates *Plücker coordinates* of W, and by definition they are defined up to scalar multiplication (i.e. they are homogenous coordinates).

A linear algebra argument shows that the previous map is injective [2]. We will call such map the *Plücker embedding*, and with it we can identify $G(k,n)$ with its image $p(G(k,n))$. We can also use the Plücker embedding to turn $G(k,n)$ into a topological space by giving it the topology making such inclusion a homeomorphism onto its image. We will usually denote this image also by $G(k,n)$, always thinking of it as living in projective space.

The next thing we will do is showing that $G(k,n)$ is not just a subset of $\mathbb{P}^N$ but has a much richer structure: it is an analytic manifold and an irreducible algebraic variety.

To see it is an analytic manifold, note that there is a bijection between k-dimensional subspaces in the open set $p_{i_1,\dots,i_k} \neq 0$ and $\mathbb{A}^{k(n-k)}$, taking a subspace W to the unique $k \times (n-k)$ matrix B such that

$$\begin{pmatrix} 1 & \dots & 0 & b_{1,k+1} & \dots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & b_{k,k+1} & \dots & b_{kn} \end{pmatrix} \qquad \text{(case } i_j = j\text{)}$$

is a matrix representation of W. The transition maps are given by matrix multiplication and thus are polynomial, so that $G(k,n)$ is an analytic manifold.

To prove it is an algebraic variety we will use the exterior algebra $\wedge^\bullet V$ associated to a vector space V [6]. Recall we say an element $\eta \in \wedge^k V$ is *decomposable* if there exist $v_1,\dots,v_k \in V$ such that $\eta = v_1 \wedge \dots \wedge v_k$. Seeing the Plücker embedding as the map

$$\langle w_1,\dots,w_k\rangle \in G(k,n) \mapsto [w_1 \wedge \dots \wedge w_k] \in \mathbb{P}(\wedge^k V) = \mathbb{P}^{\binom{n}{k}-1},$$

the image of the Grassmannian consists of (equivalence classes of) decomposable elements in $\wedge^k V$. Conversely, if $0 \neq \eta = v_1 \wedge \dots \wedge v_k$ is a decomposable multivector then $[\eta] = p(\langle v_1,\dots,v_k\rangle)$, where $v_1,\dots,v_k$ are linearly independent because $\eta \neq 0$. Therefore, characterizing the Grassmannian in $\mathbb{P}^N$ boils down to characterizing decomposable multivectors. Given $\eta \in \wedge^k V$, consider the linear map

$$\begin{aligned} \wedge\eta : V &\to \wedge^{r+1} V \\ v &\mapsto v \wedge \eta \end{aligned}$$

The following is a well-known fact from linear algebra:

Lemma 3.1.2. *Let $0 \neq \eta \in \wedge^k V$ be a non-zero multivector. Then η is decomposable if and only if* $\dim(\ker \wedge\eta) \geq k$.

Since $\dim(\ker \wedge\eta) \geq k$ if and only if $\text{rk} \wedge\eta \leq n-k$, we find that

$$p(G(k,n)) = \{\eta \in \wedge^k V \mid \text{rk} \wedge\eta \leq n-k\}.$$

This defines $p(G(k,n))$ as the zero-locus of a set of homogenous polynomials of degree $n-k+1$ corresponding to the $(n-k+1)$-minors of a matrix associated to the linear map $\wedge\eta$. This shows the Grassmannian is an algebraic variety.

Lastly, to see it is irreducible we consider the *incidence correspondence*

$$\Gamma = \{(p,W) \in \mathbb{P}^n \times \mathbb{G}(k,n) \,|\, p \in W\} \subset \mathbb{P}^n \times \mathbb{G}(k,n)$$

with projections $p_1 : \Gamma \to \mathbb{P}^n$ and $p_2 : \Gamma \to \mathbb{G}(k,n)$. Note that Γ is a projective variety, as $[v] \in \mathbb{P}(W) = \mathbb{P}(\langle w_0, \ldots, w_k \rangle)$ if and only if

$$v \wedge w_0 \wedge \ldots \wedge w_k = 0.$$

Furthermore, p_1 is surjective and regular, and its fibers are k-subspaces through a point, so irreducible $\mathbb{P}^{n-k}$'s. By Theorem 1.3.4, Γ is irreducible and thus so is $\mathbb{G}(k,n) = p_2(\Gamma)$.

3.2 The Chow ring of $\mathbb{G}(k,n)$

In this section we will work out the ring structure of the Chow ring of the Grassmannian. We start by giving some useful definitions:

Definition 3.2.1. A *complete flag* $\mathcal{V}$ on a vector space V of dimension n is a sequence of subspaces

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

with $\dim V_i = i$.

Note that any basis $\{e_1, \ldots, e_n\}$ of V gives a complete flag by taking $V_i = \langle e_1, \ldots, e_i \rangle$, and similarly every complete flag gives a (non-canonical) basis.

Definition 3.2.2. Given a complete flag $\mathcal{V}$ and a sequence of integers $a = (a_1, \ldots, a_k)$ with

$$n-k \geq a_1 \geq \cdots \geq a_k \geq 0,$$

we define the *Schubert cycle* $\Sigma_a \subset G(k,n)$ by

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G(k,n) \,|\, \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \text{ for all } i\}$$

We will often write Σ_a whenever the complete flag $\mathcal{V}$ is either known or irrelevant, we will suppress trailing zeros and write $\Sigma_{a_1 \ldots a_r}$ for $\Sigma_{a_1 \ldots a_r 0 \ldots 0}$, and we will use the notation Σ_{i^k} for the Schubert cycle $\Sigma_{i \ldots i}$ with the first k indices equal to i.

To understand the notation, consider the sequence

$$0 \subseteq V_1 \cap \Lambda \subseteq \ldots \subseteq V_{n-1} \cap \Lambda \subseteq V_n \cap \Lambda = \Lambda$$

If Λ is a general k-plane, then $V_i \cap \Lambda = 0$ for $i \leq n-k$ and $\dim(V_{n-k+i} \cap \Lambda) = i$ for $i > 0$. In other words, the dimension is constant and equal to zero for the first $n-k$ cases and then jumps one in each step. Then, we see the Schubert cycle $\Sigma_a(\mathcal{V})$ consists of those k-planes Λ for which the i-th jump occurs at least a_i steps early.

Example 3.2.3. The following are some important examples that will be used:

- $\Sigma_{n-k+1-l}$ is the set of k-planes meeting a given subspace of dimension l. In particular, Σ_1 is the set of k-planes meeting a given subspace of complementary dimension $n-k$.
- $\Sigma_{(n-l)^k}$ is the set of k-planes contained in a given l-dimensional one.
- $\Sigma_{(n-k)^r}$ is the set of k-planes that contain a given r-dimensional one.

The first important thing to notice is that the affine group GL_n acts transitively on the set of complete flags. Therefore, by the third part of Kleiman's Theorem 2.3.1, the class in $A(G(k,n))$ defined by a Schubert cycle $\Sigma_a(\mathcal{V})$ depends only on the sequence a and not on the specific flag $\mathcal{V}$. This makes sense of the following definition:

Definition 3.2.4. Given a sequence a, we call the class

$$\sigma_a := [\Sigma_a(\mathcal{V})] \in A(G(k,n))$$

for any flag $\mathcal{V}$ a *Schubert class*.

The strategy to calculate the Chow ring of the Grassmannian will be to construct an affine stratification and apply Theorem 3. The Schubert cells will turn out to be the open strata and therefore the Chow ring will be generated by the Schubert classes.

For this, define a partial order on the set of sequences as $(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k)$ if $a_i \leq b_i$ for all $i = 1, \ldots, k$. Furthermore, let $a < b$ whenever $a \leq b$ but $a \neq b$, i.e. whenever there exists an index i for which $a_i < b_i$. By definition it is clear that $\Sigma_a \subseteq \Sigma_b$ if and only if $a \geq b$, and $\Sigma_a \subsetneq \Sigma_b$ precisely when $a > b$. Thus, this partial order on the set of sequences resembles the partial order in the Schubert cycles given by inclusion. Define the *Schubert cell* Σ_a° to be

$$\Sigma_a^\circ := \Sigma_a \setminus \bigcup_{b>a} \Sigma_b.$$

If we write $|a| = \sum_{i=1}^{k} a_i$ the claim is the following:

Lemma 3.2.5. *The Schubert cells form an affine stratification of the Grassmannian. In particular, $\Sigma_a^\circ \cong \mathbb{A}^{k(n-k)-|a|}$ is smooth and irreducible and Σ_a is irreducible and of codimension $|a|$ in $G(k,n)$.*

Proof. By definition, it is clear that $G(k,n)$ is a disjoint union of the Schubert cells, and that each Schubert cycle can be written as a union of Schubert cells.

To prove that $\Sigma_a^\circ \cong \mathbb{A}^{k(n-k)-|a|}$, let $V_i = \langle e_1, \ldots, e_i \rangle$. Given a subspace $\Lambda \in \Sigma_a$, consider the sequence

$$0 \subseteq V_1 \cap \Lambda \subseteq \ldots \subseteq V_n \cap \Lambda = \Lambda.$$

As $\Lambda \in \Sigma_a$ we have $\dim(V_{n-k+1-a_1} \cap \Lambda) \geq 1$;[1] let then $v_1 \in V_{n-k+1-a_1} \cap \Lambda$. Similarly, $\dim(V_{n-k+2-a_2} \cap \Lambda) \geq 2$, so that we can choose a vector $v_2 \in V_{n-k+2-a_2} \cap \Lambda$ independent from v_1. Continuing this way, we obtain a basis $v_1, \ldots, v_k$ for Λ such that $v_i \in V_{n-k+i-a_i}$. In terms of this basis and the basis $e_1, \ldots, e_n$ for V, Λ can be represented by a matrix of the form

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \end{pmatrix} \qquad (\text{case } k = 4, n = 9, a = (3,2,2,1)).$$

If Λ was a general k-subspace and we chose its basis in the same manner, by the argument about the increasing dimensionality of the sequence of intersections $V_i \cap \Lambda$, the corresponding matrix would be of the form

$$\begin{pmatrix} * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 \\ * & * & * & * & * & * & * & * & * \end{pmatrix}$$

Thus, we see that in the i-th row we are eliminating a_i degrees of freedom, and therefore $\Sigma_a^\circ \cong \mathbb{A}^{k(n-k)-|a|}$ as wanted. □

We now know by Theorem 3 that $A(G(k,n))$ is generated as an abelian group by the Schubert classes. To find the multiplicative structure, we will adopt the following strategy:

[1] In fact, since $\Lambda \in \Sigma_a^\circ$ we have the equality, and the intersection is zero for smaller V_i's.

1. Obtain a formula for the product $\sigma_a\sigma_b$, where $a = (a_1, \ldots, a_k)$ is any sequence and b is a natural number with $0 \leq b \leq n-k$.[2] This will be the purpose of Pieri's formula (Proposition 3.2.6).
2. Find a way to express every class in terms of special Schubert classes. This will be the purpose of Giambelli's formula (Proposition 3.2.8).
3. Given any two classes, write one in terms of special Schubert classes (apply Pieri) and then use Giambelli to obtain the product.

The two main results just mentioned are the following:

Proposition 3.2.6 (Pieri's formula)**.** *Let $\sigma_b = \sigma_{b,0,\ldots,0}$ be a special Schubert class. Then, for any Schubert class $\sigma_a \in A(G(k,n))$ we have*

$$\sigma_a\sigma_b = \sum_{\substack{|c|=|a|+b \\ a_i \leq c_i \leq a_{i-1}\,\forall i}} \sigma_c$$

Proof. First note that $\sigma_a\sigma_b \in A^{|a|+b}(G(k,n))$, so that we can write

$$\sigma_a\sigma_b = \sum_{|c|=|a|+b} \gamma_c\sigma_c$$

for some coefficients $\gamma_c \in \mathbb{Z}$. We must show that these coefficients are either zero or one, and that the latter happens precisely when $a_i \leq c_i \leq a_{i-1}$ for every i.

The key for this proof is that for every $\sigma_c \in A^{|c|}(G(k,n))$ there is a unique Schubert class $\sigma_{c*} \in A^{k(n-k)-|c|}(G(k,n))$ of complementary dimension such that $\sigma_c\sigma_{c^*} = 1$ and $\sigma_c\sigma_d = 0$ for any other $\sigma_d \in A^{k(n-k)-|c|}$. In this sense, we obtain that the generators of $A^r(G(k,n))$ and of $A^{k(n-k)-r}(G(k,n))$ are mutually dual under the pairing

$$A^r(G(k,n)) \times A^{k(n-k)-r}(G(k,n)) \to A^{k(n-k)} \cong \mathbb{Z}.$$

(Note that $A^{k(n-k)}$ consists of points, and all are rationally equivalent, so $A^{k(n-k)} \cong \mathbb{Z}$.) More precisely, we are stating the following Proposition:

Proposition 3.2.7. *For classes of σ_a and σ_b of complementary dimension (i.e. $|a| + |b| = k(n-k)$), we have*

$$\sigma_a\sigma_b = \begin{cases} 1 & \text{if } a_i + b_{k+1-i} = n-k \text{ for all } i \\ 0 & \text{else} \end{cases}$$

[2]The classes of the form σ_b are called *special Schubert classes.*

Assume for now this is true (a proof can be found at the end). We will define $a^* := (n-k-a_k, \dots, n-k-a_1)$ to be the *dual index* of $a = (a_1, \dots, a_k)$. Then we can find the coefficients γ_c as

$$\gamma_c = \deg(\sigma_a \sigma_b \sigma_{c^*})$$

where the degree homomorphism $\deg : A^{k(n-k)}(G(k,n)) \to \mathbb{Z}$ counts points. We have to show that

$$\deg(\sigma_a \sigma_b \sigma_{c^*}) = \begin{cases} 1 & \text{if } a_i \le c_i \le a_{i-1} \text{ for all } i \\ 0 & \text{else} \end{cases} .$$

The sketch of the proof is as follows (see [5] for more detail):

1. Show that $\deg(\sigma_a \sigma_b \sigma_{c^*}) = 0$ if $c_i < a_i$ for some i. For this, consider general flags $\mathcal{V}$ and $\mathcal{W}$ and let

$$A_i := V_{n-k+i-a_i} \cap W_{k+1-i+c_i}.$$

   It can be shown using generality of the flags that either $A_i = 0$ or $\dim A_i = c_i - a_i + 1$. By definition of the Schubert cycles $\Sigma_a(\mathcal{V})$ and $\Sigma_{c^*}(\mathcal{W})$, an element $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ will satisfy $\Lambda \cap A_i \neq 0$, so in particular $A_i \neq 0$. Thus, if $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \neq \emptyset$ then $c_i \geq a_i$. Equivalently, if $c_i < a_i$ then the intersection is empty and $\deg(\sigma_a \sigma_b \sigma_{c^*}) = 0$.

2. Show that $\deg(\sigma_a \sigma_b \sigma_{c^*}) = 0$ if $c_i > a_{i-1}$ for some i. With some linear algebra one can show that if $A = \langle A_1, \dots, A_k \rangle$ then

$$\dim A \le k + b,$$

   with equality holding if and only if $c_i \le a_{i-1}$. Given another general flag $\mathcal{U}$, if there existed an element $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$, then by definition this requires $A \cap U_{n-k+1-b} \neq 0$. Being $U_{n-k+1-b}$ general, this implies $\dim A \geq k + b$ and thus $\dim A = k + b$, which is equivalent to $c_i \le a_{i-1}$. Hence, if $c_i > a_{i-1}$ for some i then $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ is empty, i.e. $\deg(\sigma_a \sigma_b \sigma_{c^*}) = 0$.

3. Show that if the condition $a_i \le c_i \le c_{i-1}$ holds for all i, then the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ consists of a single element. For this note that $\dim A = k + b$ and $\operatorname{codim} U = k + b - 1$, so $A \cap U$ is one-dimensional. If v is any vector in this intersection, the fact that $A = \oplus_{i=1}^k A_i$ (which follows from $\dim A = k + b$) lets us write

$$v = v_1 + \cdots + v_k \quad \text{with } v_i \in A_i.$$

Generality of the flags implies $\Lambda \subset A$ for any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$. If furthermore $\Lambda \in \Sigma_b(\mathcal{U})$, then one can see that $v_1, \dots, v_k \in \Lambda$. Thus, the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ consists of a unique point, namely the subspace $\langle v_1, \dots, v_k \rangle$.

□

Proof of Proposition 3.2.7. Consider two general flags $\mathcal{V}$ and $\mathcal{W}$, such that by Kleiman's theorem the intersection of the corresponding Schubert cycles $\Sigma_c(\mathcal{V})$ and $\Sigma_d(\mathcal{V})$ is generically transverse.[3] Then we have

$$\deg \sigma_c \sigma_d = \#(\Sigma_c(\mathcal{V}) \cap \Sigma_d(\mathcal{W})).$$

First thing we will do is prove that if $\Sigma_c(\mathcal{V}) \cap \Sigma_d(\mathcal{W}) \neq \emptyset$ then $c_i + d_{k-i+1} \leq n-k$ for all i. Consider the i-th and the $(k-i+1)$-th condition associated to the flags $\Sigma_c(\mathcal{V})$ and $\Sigma_d(\mathcal{W})$ respectively, namely

$$\dim(\Lambda \cap V_{n-k+i-c_i}) \geq i \quad \text{and} \quad \dim(\Lambda \cap W_{n-i+1-d_{k-i+1}}) \geq k-i+1.$$

Given $\Lambda \in \Sigma_c(\mathcal{V}) \cap \Sigma_d(\mathcal{W})$ satisfying both conditions, the intersection

$$(\Lambda \cap V_{n-k+i-c_i}) \cap (\Lambda \cap W_{n-i+1-d_{k-i+1}})$$

will be non-zero, as the sum of their dimensions will be at least $k+1 > \dim \Lambda$. In particular $V_{n-k+i-c_i} \cap W_{n-i+1-d_{k-i+1}} \neq 0$, so by generality of the flags

$$n-k+i-c_i+n-i+1-d_{k-i+1} \geq n+1,$$

or equivalently $c_i + d_{k-i+1} \leq n-k$.

Next thing will be to show that if the intersection is non-empty then the equalities hold. For this we use that the cycles have complementary codimension, so that

$$k(n-k) = |c| + |d| = \sum_{i=1}^{n} (c_i + d_{k+1-i}).$$

Since $c_i + d_{k+1-i} \leq n-k$ for all i, the only possible way for the sum to equal $k(n-k)$ is that

$$c_i + d_{k+1-i} = n-k \quad \text{for all } i.$$

[3]In fact, the intersection is transverse, as it is generically transverse and zero-dimensional

Until now we have shown that $\deg(\sigma_c\sigma_d) = 0$ whenever some equality $c_i + d_{k+1-i} = n - k$ does not hold. If equality holds for all i then the subspaces $V_{n-k+i-c_i}$ and $W_{n-i+1-d_{k-i+1}}$ meet in a one-dimensional subspace Γ_i, which must be contained in any $\Lambda \in \Sigma_c(\mathcal{V}) \cap \Sigma_d(\mathcal{W})$. Thus, the intersections contains a unique point, namely $\Lambda = \Gamma_1 \oplus \cdots \oplus \Gamma_k$. □

Our second main result, Giambelli's formula, is actually a Corollary of Pieri's:

Proposition 3.2.8 (Giambelli's formula)**.**

$$\sigma_{a_1,\dots,a_k} = \begin{vmatrix} \sigma_{a_1} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \cdots & \sigma_{a_2+k-2} \\ \vdots & \ddots & \vdots \\ \sigma_{a_k-k+1} & \cdots & \sigma_{a_k} \end{vmatrix}$$

Proof. The proof is by induction on k: one assumes the result for $k-1$, then expands the determinant along the righ-hand column and applies Pieri. We will show the idea by proving the case $k = 3$ assuming the case $k = 2$ holds (which is an easy consequence of Pieri's formula, as

$$\begin{aligned}\begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} \\ \sigma_{a_2-1} & \sigma_{a_2} \end{vmatrix} &= \sigma_{a_1}\sigma_{a_2} - \sigma_{a_2-1}\sigma_{a_1+1} \\ &= (\sigma_{a_1a_2} + \sigma_{a_1+1,a_2-1} + \cdots + \sigma_{a_1+a_2}) - (\sigma_{a_1+1,a_2-1} + \cdots + \sigma_{a_1+a_2}) \\ &= \sigma_{a_1a_2}\end{aligned}$$

where we have simply used Pieri's formula in the second equality). The calculation for $k = 3$ is as follows

$$\begin{aligned}\begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} \end{vmatrix} &= \sigma_{a_1+2}\begin{vmatrix} \sigma_{a_2-1} & \sigma_{a_2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} \end{vmatrix} - \sigma_{a_2+1}\begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} \\ \sigma_{a_3-2} & \sigma_{a_3-1} \end{vmatrix} \\ &\quad + \sigma_{a_3}\begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} \\ \sigma_{a_2-1} & \sigma_{a_2} \end{vmatrix} \\ &= \sigma_{a_1+2}\sigma_{a_2-1,a_3-1} - \sigma_{a_2+1}\sigma_{a_1,a_3-1} + \sigma_{a_3}\sigma_{a_1,a_2} \\ &= \sum_{\substack{|a|=|c| \\ (*)}} \sigma_{c_1c_2c_3} - \sum_{\substack{|a|=|c| \\ (**)}} \sigma_{c_1c_2c_3} + \sum_{\substack{|a|=|c| \\ (***)}} \sigma_{c_1c_2c_3}\end{aligned}$$

where

$$\begin{aligned}(*) &\equiv (c_3 \le a_3 - 1 \le c_2 \le a_2 - 1 \le c_1) \\ (**) &\equiv (c_3 \le a_3 - 1 \le c_2 \le a_2 + 1 \le c_1) \\ (***) &\equiv (c_3 \le a_2 \le c_2 \le a_1 \le c_1)\end{aligned}$$

Examining these inequalities carefully one sees that all the terms except $\sigma_{a_1a_2a_3}$ cancel. □

With Pieri's and Giambelli's formula, in theory we have an algorithm to compute any product in the Chow ring. This is feasible for small k, but becomes increasingly harder as k increases. Other methods include pictorial calculation with the so called *Young diagrams* or specific formulas for the Grassmannian of lines $\mathbb{G}(1, n)$ [5]. Note that as a Corollary of Giambelli's formula the Chow ring of the Grassmannian is generated as a ring by the special Schubert cycles.

4 Chern classes

This section will give a brief introduction to Chern classes, which are a powerful tool for computations in intersection theory. We will use them in the next section to calculate the number of lines on a cubic surface . For a more detailed exposition we refer to [5], and for concepts in algebraic geometry such as vector bundles or divisors to [3].

4.1 The first Chern class of a line bundle

We will begin by motivating the first Chern class and then we will introduce Chern classes in general axiomatically. Fix a line bundle $\mathcal{L}$ and let τ be a rational section of $\mathcal{L}$. On an affine open $U \subseteq X$ we can write τ as f_U/g_U and define

$$\mathrm{Div}(\tau)|_U := \mathrm{Div}(f) - \mathrm{Div}(g).$$

Note that if $V \subseteq X$ is another affine, then $\mathrm{Div}(\tau)|_U$ and $\mathrm{Div}(\tau)|_V$ agree on $U \cap V$, so that we get a divisor $\mathrm{Div}(\tau)$ on X. If τ' is another rational section of $\mathcal{L}$ then τ/τ' is a well-defined rational function, and thus

$$\mathrm{Div}(\tau) - \mathrm{Div}(\tau') = \mathrm{Div}(\tau/\tau') \in \mathrm{Rat}(X).$$

Therefore, the following definition makes sense:

Definition 4.1.1. For a line bundle $\mathcal{L}$, we define its *first Chern class* to be

$$c_1(\mathcal{L}) = [\mathrm{Div}\,\tau] \in A_{n-1}(X)$$

for any rational section τ of $\mathcal{L}$.

It is easy to see that the assignment

$$c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$$

taking each isomorphism class of line bundles to its Chern class is a group homomorphism, i.e. $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$ (if τ and τ' are sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, consider the rational section $\tau \otimes \tau'$ of $\mathcal{L} \otimes \mathcal{L}'$ with divisor $\mathrm{Div}(\tau) + \mathrm{Div}(\tau')$). We also see that if $c_1(\mathcal{L}) = 0$ then $\mathcal{L}$ has a nowhere vanishing section, and therefore $\mathcal{L}$ is trivial.

4.2 Chern classes of vector bundles

We have just defined the first Chern class of a line bundle, and seen it measures its triviality. We will now define the i-th Chern class of a vector bundle of any rank: the definition for the first Chern class will follow in a similar manner, and the i-th Chern class will be defined axiomatically.

For a line bundle $\mathcal{L}$ we defined $c_1(\mathcal{L}) = [\mathrm{Div}\,\tau] \in A_{n-1}(X)$ for any rational section τ, and we now declare $c_i(\mathcal{L}) = 0$ for $i > 1$. Given a vector bundle $\mathcal{E}$ of rank r, the first Chern class $c_1(\mathcal{E})$ will as well aim to detect triviality of $\mathcal{E}$, in the sense that if $\mathcal{E}$ is trivial then $c_1(\mathcal{E}) = 0$. For this, note that $\mathcal{E}$ is trivial if and only if there exist r (everywhere) independent global sections $s_0, \ldots, s_{r-1}$, and that if such sections exist then any collection of r general sections will also be independent. By the defining properties of the exterior algebra, the locus where r sections $s_0, \ldots, s_{r-1}$ become linearly dependent will be precisely the zero locus of the section $s_0 \wedge \ldots \wedge s_{r-1}$ of the bundle $\wedge^r \mathcal{E}$. As the rank of $\mathcal{E}$ is r, the new vector bundle $\wedge^r \mathcal{E}$ will be of rank one, so by the previous discussion the vanishing locus of $s_0 \wedge \ldots \wedge s_{r-1}$ will be

$$c_1(\mathcal{E}) := c_1(\wedge^r \mathcal{E}) \in A_{n-1}(X),$$

which we call the *first Chern class of* $\mathcal{E}$.

To characterize Chern classes we use the following Theorem:

Theorem 4.2.1. *There is a unique way of assigning to each rank r vector bundle $\mathcal{E}$ on a smooth quasi-projective variety X a class*

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X)$$

with $c_i(\mathcal{E}) \in A^i(X)$ in such a way that:

1. *(Line bundles) If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$, where $c_1(\mathcal{L}) \in A^1(X)$ is the class of the divisor of zeros minus the divisor of poles of any rational section of $\mathcal{L}$.*

2. *(Bundles with enough sections) If* $s_0, \dots, s_{r-i}$ *are global sections of* $\mathcal{E}$*, and the degeneracy locus* D *where they are dependent has codimension* i*, then* $c_i(\mathcal{E}) = [D] \in A^i(X)$*.*
3. *(Whitney's formula) If*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is a short exact sequence of vector bundles on X *then*

$$c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'')$$

4. *(Functoriality) If* $\varphi : Y \to X$ *is a morphism of smooth varieties, then*

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E}))$$

The second part of this Theorem says that the i-th Chern class is the locus where $r - i + 1$ general sections $s_0, \dots, s_{r-i}$ become linearly dependent. As before, this will be the vanishing locus of the section $s_0 \wedge \dots \wedge s_{r-i}$ of the bundle $\wedge^{r-i+1}\mathcal{E}$, and we call it the *degeneracy locus* of the sections $s_0, \dots, s_{r-i}$.

To get a grasp of these classes, consider the simple case where we have a rank k vector bundle $\mathcal{E}$ over $\mathbb{P}^n$ that splits as a direct sum of line bundles, i.e.

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)$$

where we take $d_i > 0$ for simplicity. Then

$$c(\mathcal{O}_{\mathbb{P}^n}(d_i)) = 1 + [D_i] = 1 + d_i[H]$$

for D_i a degree d_i hypersurface, so using Whitney's formula we see that the k-th Chern class of $\mathcal{E}$ is

$$c_k(\mathcal{E}) = [D_1] \dots [D_k],$$

which is a complete intersection of type $(d_1, \dots, d_k)$. Similarly, the $(k-m)$-th Chern class will be

$$c_{k-m}(\mathcal{E}) = \sum_{1 \le i_1 < \dots < i_{k-m} \le k} [D_{i_1}] \dots [D_{i_{k-m}}]$$

which is a sum of complete intersections of type $(d_{i_1}, \dots, d_{i_{k-m}})$. In this way, we can also see the first Chern class of $\mathcal{E}$ as

$$c_1(\mathcal{E}) = [D_1] + \dots + [D_k] = (d_1 + \dots + d_k)[H]$$

which is just a degree $d_1 + \cdots + d_k$ hypersurface.

The key to make the previous example so simple was the fact that the bundle $\mathcal{E}$ split as a direct sum of line bundles, whose Chern classes are known and occur only in degree one. Of course, not every vector bundle splits like this. However, the following result says this is actually how we should think about Chern classes in general:

Theorem 4.2.2 (Splitting principle)**.** *If an identity among Chern classes is true for bundles that split as direct sum of line bundles, then it is true for any vector bundle.*

To prove this one constructs a space Y and a morphism $\varphi : Y \to X$ such that the pullback map $\varphi^* : A(X) \to A(Y)$ is injective and the pulled back bundle $\varphi^*(\mathcal{E})$ admits a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \mathcal{E}_r = \varphi^*(\mathcal{E})$$

with each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ a line bundle. Then the functoriality property 4 gives the result.

The splitting principle is the key ingredient we will use in the next section to compute the Chern class of the (dual of the) universal subbundle of the Grassmannian.

5 Applications

This section will be devoted to solving two classical problems in intersection theory. The first will be to count the number of lines in $\mathbb{P}^3$ meeting four given lines in general position, as well as a higher-dimensional generalization. The second will be to find the lines contained in a cubic surface. For this section we refer to [4] and [5].

5.1 Lines meeting four general lines in $\mathbb{P}^3$

We will answer this classical problem and its generalization in two different ways, one algebraic and the other geometric.

We start with the algebraic approach, which will turn out to be straightforward after the development in Section 3.2. Given a line L in $\mathbb{P}^3$, the set of lines meeting L is just the Schubert cycle

$$\Sigma_1(L) = \{\Lambda \in \mathbb{G}(1,3) \,|\, \Lambda \cap L \neq \emptyset\}.$$

Therefore, the set of lines meeting four lines $L_1, \dots, L_4$ will be $\cap_{i=1}^4 \Sigma_1(L_i)$. If the L_i are general, by Kleiman's transversality the Schubert cells $\Sigma_1(L_i)$ will intersect generically transversely, so that

$$\#\left(\cap_{i=1}^4 \Sigma_1(L_i)\right) = \deg \sigma_1^4.$$

To compute the degree of this class, we use the proposed strategy. Using Pier's formula we obtain

$$\sigma_1\sigma_1 = \sigma_2 + \sigma_{11}$$

and therefore

$$\sigma_1^4 = \sigma_2^2 + \sigma_{11}^2 + 2\sigma_2\sigma_{11}.$$

We calculate each term independently:

1. The first term equals σ_{22} by Pieri.
2. For the second, we use Giambelli to get

   $$\sigma_{11} = \begin{vmatrix} \sigma_1 & \sigma_2 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_1^2 - \sigma_2.$$

   By Pieri $\sigma_1^2\sigma_{11} = \sigma_1\sigma_{21} = \sigma_{22}$ and similarly $\sigma_2\sigma_{11} = 0$. Therefore $\sigma_{11}^2 = \sigma_{22}$.
3. Here we obtain $\sigma_2\sigma_{11} = 0$.

Putting everything together, the answer to our problem is

$$\#(\cap_{i=1}^4 \Sigma_1(L_i)) = 2,$$

i.e. there are exactly two lines meeting four lines in general position.

Now we present a geometric solution to the problem which does not need Schubert calculus and uses some beautiful geometric constructions. Consider initially three general lines L_1, L_2, L_3, which will be skew by generality. The strategy will be to build a ruled surface containing these three lines and then use Bezout's theorem.

Lemma 5.1.1. *Given any point $p \in L_1$, there exists a unique line L_p passing through p and intersecting both L_2 and L_3.*

Proof. Choose a general hyperplane $H \cong \mathbb{P}^2$. If we project L_2 and L_3 from p to H we obtain two lines L_1' and L_2' contained in H (here we are using H is general). By generality $L_1' \neq L_2'$, so there exists a unique point $p' \in L_1' \cap L_2'$. It is then clear that the line $L_p := \overline{pp'}$ is the unique line through p intersecting both L_2 and L_3. □

Lemma 5.1.2. *Given two different points $p_1, p_2 \in L_1$, we have $L_{p_1} \cap L_{p_2} = \emptyset$.*

Proof. Suppose for contradiction that there exists $x \in L_{p_1} \cap L_{p_2}$. Then L_{p_1} and L_{p_2} lie in the plane spanned by p_1, p_2 and x, and in particular so do L_1, L_2 and L_3. Then the three lines would be coplanar, contradicting the generality hypothesis. $\square$

Lemma 5.1.3. *There exists a unique quadric containing the three lines L_1, L_2 and L_3. Moreover, this quadric is just*

$$Q = \bigsqcup_{p \in L_1} L_p.$$

Proof. Consider the restriction maps

$$r_i : H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathcal{O}_{L_i}(2)), \quad i = 1, 2, 3.$$

Each line $L_i \cong \mathbb{P}^1$ has a three-dimensional family of quadratic polynomials, i.e. $\dim H^0(\mathcal{O}_{L_i}(2)) = 3$, and similarly $\dim H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 10$. Since the restriction maps are linear we deduce $\dim(\ker r_i) \geq 3$, and thus

$$\dim \left(\bigcap_{i=1}^{3} \ker r_i \right) \geq 1$$

by Grassmann's formula. Therefore there is at least one quadratic polynomial in $\mathbb{P}^3$ vanishing on the three lines, i.e. there is at least one quadric Q containing the three lines.

For uniqueness, take a line L_p. Since it meets the three lines at different points and the lines are contained in Q, we see L_p meets Q at at least three points, so by Bézout's Theorem $L_p \subset Q$. As the union of the lines L_p is a non-degenerate surface (for $p \neq p'$, the two lines L_p and $L_{p'}$ do not intersect), it follows that $Q = \sqcup_{p \in L_1} L_p$. $\square$

With this in mind we can easily answer the initial problem. Let L_4 be the other line. By generality it will not be contained in Q, and thus it will intersect Q at two points $q, q' \in Q$. By construction of Q these points belong to two unique lines L_p and $L_{p'}$, and these are precisely the two lines intersecting all $L_1, \ldots, L_4$.

Lastly, we show that using similar techniques it is possible to solve the following more general problem: *"how many lines meet four general n-planes*

$V_1, \dots, V_4 \subset \mathbb{P}^{2n+1}$*?"*. Again, we can answer this both algebraically and geometrically. For the former, note that the set of lines $L \subset \mathbb{P}^{2n+1}$ meeting an n-plane V is just the Schubert cycle

$$\Sigma_n(V) = \{\Lambda \in \mathbb{G}(1, 2n+1) \mid \Lambda \cap V \neq \emptyset\}.$$

Using Kleiman's transversality the answer to the problem is $\deg \sigma_n^4$, which we can calculate in a similar manner (but much more involved) as before to be

$$\deg \sigma_n^4 = n + 1,$$

which obviously agrees with the initial case $n = 1$. We mention that a much more efficient way to caluclate this product is to develop particular product formulas for the Grassmannian $\mathbb{G}(1, n)$ [5].

The geometric argument is similar as before and left as an exercise in [5]. We present a solution here, which is based on the following Lemma:

Lemma 5.1.4. *The union of the lines meeting* V_1, V_2 *and* V_3 *is a Segre variety* $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$.

Proof. Consider the Segre embedding $\Psi_{1,n} : \mathbb{P}^1 \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$ and the projective reference $P_1 = [1 : 0], P_2 = [0 : 1], P_3 = [1 : 1]$ of $\mathbb{P}^1$. The images $\Psi_{1,n}(\{P_i\} \times \mathbb{P}^n)$ of the three n-planes $\{P_i\} \times \mathbb{P}^n$ under the Segre embedding will be three general n-planes in $\mathbb{P}^{2n+1}$. The idea will be to reduce the starting case of three general n-planes to this one, where the result is tautological.

Let $\mathcal{B} = (e_0, \dots, e_{2n+1})$ be the canonical basis of k^{2n+2}. After a projective transformation, we can assume without loss of generality that V_1 and V_2 are respectively the projectivizations of the $(n+1)$-planes $\langle e_0, \dots, e_n\rangle$ and $\langle e_{n+1}, \dots, e_{2n+1}\rangle$. Choose a basis $v_0, \dots, v_n$ for V_3. Note that if we write $v_i = (v_i^0, \dots, v_i^{2n+1})$ then the vectors $w_i = (v_i^0, \dots, v_i^n)$ are linearly independent: if they were not, we could find a linear combination of the v_i of the form

$$\lambda_0 v_0 + \cdots + \lambda_n v_n = (0, \overset{n+1}{\dots}, 0, *, \dots, *)$$

and therefore the element $\lambda_0 v_0 + \cdots + \lambda_n v_n$ would be both in V_2 and in V_3, which is a contradiction. Similarly, the vectors $(v_i^{n+1}, \dots, v_i^{2n+1})$ are linearly independent by the same argument. This means that V_3 projects isomorphically onto V_1 and V_2, so in particular the projections of the vectors v_i onto V_1 and V_2 form a basis for k^{2n+2}. The basis change from $\mathcal{B}$ to this new basis is given (with respect to $\mathcal{B}$) by the invertible matrix

$$\begin{pmatrix} v_0^0 & \dots & v_n^0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_0^n & \dots & v_n^n & 0 & \dots & 0 \\ 0 & \dots & 0 & v_0^{n+1} & \dots & v_n^{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & v_0^{2n+1} & \dots & v_n^{2n+1} \end{pmatrix}.$$

Consider the inverse of such automorphism. By construction, the induced homography carries V_1 and V_2 into themselves, and the image of V_3 has a basis of the form $e_i + e_{i+n+1}$ for $i = 0, \dots, n$. In this new configuration it is clear that the image $S_{1,n}$ of the Segre embedding $\Psi_{1,n} : \mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^{2n+1}$ given by

$$([\lambda : \mu], [y_0 : \dots : y_n]) \mapsto [\lambda y_0 : \dots : \lambda y_n : \mu y_0 : \dots : \mu y_n]$$

is the set of lines meeting the three new n-planes. Indeed, for every fixed $y \in \mathbb{P}^n$ the restriction of the map to $\mathbb{P}^1 \times \{y\}$ defines the line in $\mathbb{P}^{2n+1}$ joining the points $[y : 0 : \dots : 0] \in V_1$ and $[0 : \dots : 0 : y] \in V_2$: it intersects V_1 at $[1 : 0]$, V_2 at $[0 : 1]$ and V_3 at $[1 : 1]$. Conversely, if l is a line intersecting the three planes, let p_1 and p_2 be the points of intersection with V_1 and V_2 respectively. These two points determine l, and since l intersects V_3 there exists $[\lambda : \mu] \in \mathbb{P}^1$ such that

$$\lambda p_1 + \mu p_2 = [y_0 : \dots : y_n : y_0 : \dots : y_n].$$

This means that, up to a global factor, the first $n + 1$ coordinates of p_1 coincide with the last $n + 1$ of p_2. Thus $l = \Phi(\mathbb{P}^1 \times [y_0 : \dots : y_n])$.

As this simplified situation differs from the initial one just by projective transformations, we conclude the union of all lines meeting the three general n-planes is projectively equivalent to the Segre variety $S_{1,n}$. □

Lastly, it can be seen that the degree of $S_{1,n}$ is $\deg S_{1,n} = n + 1$. As V_4 is the intersection of $n + 1$ hyperplanes in $\mathbb{P}^{2n+1}$, the degree of $S_{1,n}$ is precisely the number of intersection points with the general n-plane V_4.

5.2 Lines on a smooth cubic surface in $\mathbb{P}^3$

The purpose of this section is to prove the following result:

Theorem 5.2.1. *Every smooth cubic surface $S \subset \mathbb{P}^3$ contains 27 lines.*

Remark 5.2.2. The smoothness hypothesis is necessary, e.g. the cone over a cubic curve contains infinitely many lines.

To prove this main result, we will start by showing the following result, whose proof is well-known (see [4, 7, 8]).

Proposition 5.2.3. *Every cubic surface contains at least one line.*

Proof. A cubic polynomial in four variables has 20 coefficients. Since rescaling the coefficients by a global factor does not change the zeros of the polynomial, the space of cubic surfaces in $\mathbb{P}^3$ is parametrized by $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Gamma = \{(l, S) \in \mathbb{G}(1,3) \times \mathbb{P}^{19} \,|\, l \subset S\} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19},$$

where we identify an element $l \in \mathbb{G}(1,3)$ with the line it spans in $\mathbb{P}^3$ and an element $S \in \mathbb{P}^{19}$ with the cubic surface it represents in $\mathbb{P}^3$. Let also $\pi_1 : \Gamma \to \mathbb{G}(1,3)$ be the natural projection onto the first factor. Given $S \in \mathbb{P}^{19}$, we want to show $p^{-1}(S) \neq \emptyset$, i.e. the set of lines contained in S is non-empty. In other words, we want to show that π_2 is surjective.

The incidence correspondence Γ can be seen to be a projective variety. Consider now the other projection $\pi_2 : \Gamma \to \mathbb{G}(1,3)$, which is clearly surjective and regular. For $l \in \mathbb{G}(1,3)$, the fiber $\pi_2^{-1}(l)$ represents all cubic surfaces containing l. If l is defined by linear polynomials F and G, then any cubic surface containing l must have an equation of the form

$$fF + gG,$$

where f and g are quadratic polynomials. This is a 16-dimensional space of polynomials, and thus $\pi_2^{-1}(l) \cong \mathbb{P}^{15}$ for any $l \in \mathbb{G}(1,3)$. It follows from Theorem 1.3.4 that Γ is irreducible, and applying Theorem 1.3.3 we obtain

$$\dim \Gamma = 15 + 4 = 19.$$

Consider now the Fermat cubic F with equation $X^3+Y^3+Z^3+T^3 = 0$, which is known to contain a finite number of lines (in fact, 27 lines). In particular $\pi_2^{-1}(F) \neq \emptyset$ and $\dim \pi_2^{-1}(F) = 0$, so by Theorem 1.3.3 $\dim \pi_2(\Gamma) = 19$. Thus, $\pi_2(\Gamma) = S$ by Theorem 1.3.2. $\square$

The next step will be to find additional lines starting from this one. The idea is to consider planes containing this line and analyze their intersection with S. This is done in the following Proposition, whose proof we sketch:

Proposition 5.2.4. *Let $l \subset S$ be a line in a smooth cubic (which exists by Proposition 5.2.3). Then there exist 10 lines meeting l and contained in S.*

Proof. Consider the hyperplanes $H \cong \mathbb{P}^2$ containing l. These will intersect S at the line l plus a conic $\mathcal{C}$.

First of all, note that if $\mathcal{C}$ is degenerate it must be the union of two lines. Indeed, if it were a double line then S would be defined by a polynomial of the form

$$fF + GJ^2,$$

with f quadratic and F, G and J linear. By computing partials one sees that S would have a singular point, which contradicts smoothness.

Next, we claim there are exactly five planes for which $\mathcal{C}$ is degenerate. To see this consider the one-parameter family of planes though l. The intersection of this family with S gives a one-parameter family of conics. One such conic will be singular if and only if the discriminant of its homogenized quadratic equations vanishes. This is a degree five polynomial in $\mathbb{P}^1$, so by Proposition 1.3.1 it has exactly five roots counted with multiplicites, and using smoothness one can show all these roots are simple.

We have proven there are exactly five different roots, i.e. five different planes which intersect the cubic in l and two other different lines. □

Remark 5.2.5. Note that two lines in two different such planes are disjoint. In particular, there exist two disjoint lines contained in S.

Lastly, we prove the following Lemma, which is essentially a consequences of the work done in the previous section:

Lemma 5.2.6. *Take four disjoint lines $l_1, \ldots, l_4 \subset \mathbb{P}^3$. If they do not lie in a smooth quadric, then there exists either one or two lines intersecting all of them*

Proof. First, note that the proof of Lemma 5.1.1 shows there is a quadric containing the three lines. Furthermore, a quadric containing these three lines is necessarily smooth, as otherwise the lines could not be disjoint.

Pick now a smooth quadric Q containing the first three lines. If l_4 is not contained in Q it will intersect Q at one or two points. Through each of these points there is a unique line intersecting all the l_i, corresponding to one of the rulings of the quadric. Therefore, the number of lines will be one or two.

□

We are now in a position to prove Theorem 5.2.1:

Proof of Theorem 5.2.1. Pick two disjoint lines l and m in S. Denote by Π_i the five planes containing l and interesecting S in two more lines (l_i, l_i'). Since m is contained in S and is disjoint from l it must intersect every pair (l_i, l_i'). Furthermore, it can't intersect both l_i and l_i', as this would imply that m, l_i and l_i' are coplanar and thus $m \cap l \neq \emptyset$, a contradiction. Thus, we can assume without loss of generality that m meets $l_1 \dots, l_5$.

Label by (l_i, l_i'') the five pairs of lines meeting m. As the lines l, l_j and l_j' form a hyperplane section of S, any line contained in S must meet one of them. For $i \neq j$ the line l_i'' does not meet l nor l_j, so we deduce that l_i'' must meet l_j'.

We have now 17 lines l, m, l_i, l_i', l_i'' contained in S. We claim that for each three different lines l_i, l_j, l_k there is a unque line l_{ijk} meeting all three, and that these are precisely the only other lines contained in S. Indeed, let $n \subset S$ be another line contained in S. Note any four lines contained in S cannot lie on a quadric, as this would mean S is not irreducible. Using Lemma 5.2.6, we see that if n met four of the l_i then $n = m$ or $n = l$, a contradiction. Therefore n meets at most three of the l_i. If it met two or less, then it would have to meet at least three of the lines l_i', and a very similar argument shows n must be one of our 17 lines. Thus, n meets exactly three of the lines l_i. Conversely, take three different lines l_i, l_j, l_k and suppose without loss of generality $i = 1$. We know there are 10 lines meeting l_1, four of which are l, l_1', m and l_1''. By our previous discussion, the remaining six must meet exactly two out of the four lines $l_2, \dots, l_5$. Since there are exactly $\binom{4}{2} = 6$ possibilities, they must all occur.

Therefore, the lines contained in S are $\{l, m, l_i, l_i', l_i'', l_{ijk}\}$ and these amount to a total of

$$1 + 1 + 5 + 5 + 5 + 10 = 27.$$

□

As a Corollary of Proposition 5.2.4 we also find the following result:

Corollary 5.2.7. *Every smooth cubic in $\mathbb{P}^3$ is rational.*

Proof. Let l and m be two disjoint lines contained in S. Given a point $p \in S \setminus (l \cup m)$, there exists a unique line n_p through p meeting both l and

m. This gives a rational map

$$\begin{aligned}\Phi : S &\dashrightarrow l \times m \\ p &\mapsto (n_p \cap l, n_p \cap m).\end{aligned}$$

Conversely, given $(q, r) \in l \times m$ consider the line $\overline{qr} \subset \mathbb{P}^3$. Since there are only ten lines meeting l contained in S, most of these lines $\overline{qr}$ will not be contained in S, and by Bezout's theorem they will intersect S at three points p, q, r. This gives a rational

$$\begin{aligned}\psi : l \times m &\dashrightarrow S \\ (q, r) &\mapsto p\end{aligned}$$

which can be seen to be inverse to ϕ. Therefore S is rational. □

We have seen that it possible to solve a quite hard enumerative problem like finding lines in a smooth cubic surface just using geometric arguments. We will now use the machinery of Section 4 to solve this problem in an alternative, more advanced way. The advantage is double: one one side, the solution is shorter, and on the other, it is generalizable to other settings.

Take a line $L \in \mathbb{G}(1, 3)$ and consider the four dimensional vector space $H^0(\mathcal{O}_L(3))$ of cubic forms defined on $L \cong \mathbb{P}^1$. As L varies on $\mathbb{G}(1, 3)$, this defines a vector bundle $\mathcal{V}$ of rank four over $\mathbb{G}(1, 3)$. Given a cubic form F on $\mathbb{P}^3$, the restrictions $H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_L(3))$ of the cubic form to lines $L \in \mathbb{P}^3$ gives a section τ_F of the vector bundle $\mathcal{V}$, and the vanishing locus of this section will precisely be the set of lines contained in the cubic. By the second part of Theorem 4.2.1, we recognize this vanishing locus as the fourth Chern class of $\mathcal{V}$. Thus, our goal is to calculate $c_4(\mathcal{V})$.

For this, consider first the vector bundle $\mathcal{S}$ of rank two over $\mathbb{G}(1, 3) \cong G(2, 4)$ whose fiber at each point $L \in G(2, 4)$ is the two-dimensional subspace represented by L. We see that the vector bundle whose fiber at each point $L \in \mathbb{G}(1, 3)$ is the vector space $H^0(\mathcal{O}_L(1))$ of linear forms on L is the dual bundle $\mathcal{S}^*$ of $\mathcal{S}$. Given this, we can write $\mathcal{V} = \operatorname{Sym}^3 \mathcal{S}^*$.

Now suppose $\mathcal{S}^*$ splits as $\mathcal{S}^* = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ line bundles. Then

$$\mathcal{V} = \operatorname{Sym}^3 \mathcal{S}^* = \mathcal{L}_1^{\otimes 3} \oplus (\mathcal{L}_1^{\otimes 2} \otimes \mathcal{L}_2) \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes 2}) \oplus \mathcal{L}_2^{\otimes 3},$$

so that writing $c_1(\mathcal{L}_1) = \alpha$ and $c_1(\mathcal{L}_2) = \beta$, Whitney's formula says

$$c(\mathcal{V}) = (1 + 3\alpha)(1 + 2\alpha + \beta)(1 + \alpha + 2\beta)(1 + 3\beta). \qquad (\star)$$

Using Whitney's formula again for $\mathcal{S}^*$ we obtain

$$c(\mathcal{S}^*) = c(\mathcal{L}_1)c(\mathcal{L}_2) = 1 + \alpha + \beta + \alpha\beta$$

and therefore $c_1(\mathcal{S}^*) = \alpha + \beta$ and $c_2(\mathcal{S}^*) = \alpha\beta$. Expanding ($\star$) and gathering terms appropriately one gets

$$\begin{aligned} c(\mathcal{V}) = 1 &+ 6c_1(\mathcal{S}^*) + (11c_1(\mathcal{S}^*)^2 + 10c_2(\mathcal{S}^*)) \\ &+ (6c_1(\mathcal{S}^*)^3 + 30c_1(\mathcal{S}^*)c_2(\mathcal{S}^*)) + (18c_1(\mathcal{S}^*)^2c_2(\mathcal{S}^*) + 9c_2(\mathcal{S}^*)^2), \end{aligned}$$

from which we identify

$$c_4(\mathcal{V}) = 9(2c_1(\mathcal{S}^*)^2c_2(\mathcal{S}^*) + c_2(\mathcal{S}^*)^2).$$

By the splitting principle, this expression is valid even if $\mathcal{S}^*$ does not split.

Lastly, take a linear form $\varphi \in (\mathbb{C}^4)^*$. By restricting φ to each plane $\Lambda \in G(2,4)$ we obtain a global section of $\mathcal{S}^*$. Note that this restriction will vanish in Λ precisely when $\Lambda \subset \ker\varphi$, i.e. it will vanish in the Schubert cycle $\Sigma_{1,1}(\ker\varphi)$. Since this has codimension two, the third part of Theorem 4.2.1 tells us that $c_2(\mathcal{S}^*) = \sigma_{1,1}$. In a similar manner one sees that $c_1(\mathcal{S}^*) = \sigma_1$, so that $c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}$. Putting everything together we get

$$c_4(\mathcal{V}) = 9(2\sigma_1^2\sigma_{1,1} + \sigma_{1,1}^2) = 9(2 \cdot 1 + 1) = 27,$$

where we have used the calculation of Section 5.1 for the products $\sigma_1^2\sigma_{11}$ and σ_{11}^2. As mentioned, this method is very powerful and can be generalized to solve many similar problems: as an exercise for the reader, we propose finding the lines in a quintic threefold in $\mathbb{P}^4$.

6 Conclusions

I hope this essay has shown the power of Schubert calculus and Chern classes to deal with enumerative problems, as well as the beauty (and usefulness!) of some geometric constructions like the quadric containing three general lines.

This subject is so wide that many interesting topics and further problems had to be left out. Some natural theoretical extensions of this essay would include Young diagrams, using Chern classes to derive the polynomial relations between special Schubert classes or generalizing this theory to *flag manifolds*, whose Chow ring exhibits many similarities. In the geometric side, the technique of *specialization* – both static and dynamic – is of particular importance and reveals some interesting geometry. Leaving the Grassmannian and linearity, one could also explore intersection theory on the moduli space of curves. This subject, which in a way generalizes the work done in this essay, is very rich and has gained a lot of attention in recent years due to its connections with theoretical physics through string theory. As for problems, with Schubert calculus we could find the lines contained in the intersection of two quadrics in $\mathbb{P}^4$, the lines secant to two degree d curves in $\mathbb{P}^3$, or the degree of $G(k, n)$ as a subvariety of projective space.

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